

# A new class of (3+1) quantum geometric models

Ion I. Cotăescu

The West University of Timișoara,  
V. Pârvan Ave. 4, RO-1900 Timișoara, Romania

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## Abstract

A new family of analytically solvable quantum geometric models is proposed. The structure of the energy spectra as well as the form of the corresponding eigenfunctions are presented pointing out their main specific properties.

In the general relativity the relativistic (three-dimensional isotropic) harmonic oscillator has been defined as a free system on the anti-de Sitter static background [1]. In a previous article we have generalized this model to a family of models the metrics of which represent suitable deformations of an anti-de Sitter static metric [2]. These are the spherically symmetric static metrics given by the line elements which can be written in spherical coordinates and natural units ( $\hbar = c = 1$ ) as [3]

$$ds^2 = \frac{\alpha}{\beta} dt^2 - \frac{\alpha}{\beta^2} dr^2 - \frac{r^2}{\beta} (d\theta^2 + \sin^2 \theta d\phi^2), \quad (1)$$

where the functions

$$\alpha = 1 + (1 + \lambda)\omega^2 r^2, \quad \beta = 1 + \lambda\omega^2 r^2 \quad (2)$$

depend on the real parameter  $\lambda$ . We have shown [3] that the quantum models of the free massive scalar particles on these backgrounds are of two kinds.

The first set contains the models with  $\lambda < 0$ . The particle of a such a model is confined to a spherical cavity having a countable energy spectrum with a fine-structure due to a rotator-like term. In the particular case of  $\lambda = -1$ , when the metric is just the anti-de Sitter one, the spectrum becomes equidistant and the fine-structure disappears [1]. The other set, corresponding to  $\lambda > 0$ , is of models with mixed spectra, having finite discrete sequences with fine-structure and continuous parts. The model with  $\lambda = 0$  is of a special interest since it is very closed to the nonrelativistic isotropic harmonic oscillator which, although, is just the nonrelativistic limit of all these models. Another characteristic of this family is that the quantum numbers which determine the discrete energy levels, the main quantum number and the angular momentum one, are separately involved in two different terms of the level formulas. For this reason these models, except the anti-de Sitter one, have been called relativistic rotating oscillators.

These results indicate that new families of analytically solvable geometric models with more sophisticated behavior could exist. By looking for them, we have found another interesting one we would like to present in the following. This new family has the metrics given by

$$ds^2 = \frac{\alpha}{\beta} dt^2 - \frac{\alpha}{\beta^2} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (3)$$

where the functions  $\alpha$  and  $\beta$  are just those defined by (2) depending on the real parameter  $\lambda$ . We note that the model with  $\lambda = 0$  is the same as that of the family defined by (1). Moreover, the general behavior with respect to  $\lambda$  of both these families is similar. Thus, we observe that when  $\lambda < 0$  the event horizon of an observer situated at  $x^i = 0$  is the sphere of the radius  $r = r_e = 1/\omega\sqrt{-\lambda}$  where the metric is singular. For non-negative  $\lambda$  this is at  $r_e = \infty$ . Here we shall consider the free motion only on the domain  $D$  of the space coordinates bounded by the event horizon, i.e.  $r \in [0, r_e)$ . Obviously, since all these metrics are invariant under time translations and space rotations, the energy and the angular momentum of the free motion are conserved.

Let  $\phi$  be a scalar massive field of the mass  $M$ , defined on  $D$ , minimally coupled with the gravitational field [4]. Because of the energy conservation, the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) + M^2 \phi = 0, \quad (4)$$

where  $g = \det(g_{\mu\nu})$ , admits the fundamental solutions

$$\phi_E^{(+)}(x) = \frac{1}{\sqrt{2E}} e^{-iEt} U(r, \theta, \phi), \quad \phi^{(-)} = (\phi^{(+)})^*, \quad (5)$$

which must be orthogonal with respect to the relativistic scalar product [4]

$$\langle \phi, \phi' \rangle = i \int_D d^3x \sqrt{-g} g^{00} \phi^*(x) \overleftrightarrow{\partial}_0 \phi'(x). \quad (6)$$

The conservation of the angular momentum allows one to separate the variables of the Klein-Gordon equation (4) by choosing particular solution (5) with

$$U(r, \theta, \phi) = R_{E,l}(r) Y_{lm}(\theta, \phi), \quad (7)$$

where  $Y_{lm}$  are the usual spherical harmonics and  $R_{E,l}$  are the radial wave functions which satisfy the radial equation

$$-\beta \frac{d^2 R_{E,l}}{dr^2} - (3\beta - 1) \frac{1}{r} \frac{dR_{E,l}}{dr} + \frac{\alpha}{r^2 \beta} l(l+1) R_{E,l} - \left( E^2 - \frac{\alpha}{\beta} M^2 \right) R_{E,l} = 0. \quad (8)$$

Moreover, from (6) it results that the scalar product reduces to the radial one,

$$\langle R, R' \rangle = \int_0^{r_e} \frac{r^2 dr}{(1 + \lambda \omega^2 r^2)^{1/2}} R^*(r) R'(r). \quad (9)$$

Now we have all the elements for deriving the energy spectra and the corresponding energy eigenfunctions. All the results in the case of  $\lambda = 0$ , which might be separately treated, are known from Ref.[3]. Therefore, we can start directly with the general case of any  $\lambda \neq 0$  where it is convenient to use the new variable  $y = -\lambda \omega^2 r^2$ , and the notations

$$\nu = \frac{1}{4\lambda\omega^2} \left[ \lambda\omega^2 + \left( 1 + \frac{1}{\lambda} \right) M^2 - E^2 \right]. \quad (10)$$

We shall look for a solution of the form

$$R(y) = N(1 - y)^{k/2} y^{l/2} F(y), \quad (11)$$

where  $k$  is a real number and  $N$  is the normalization factor. After a few manipulation we find that, for

$$k^2 - k - \frac{M^2}{\lambda^2 \omega^2} + \frac{1}{\lambda} l(l+1) = 0, \quad (12)$$

the equation (8) transforms into the following hypergeometric equation

$$y(1-y)\frac{d^2F}{dy^2} + \left[l + \frac{3}{2} - y(2p+1)\right]\frac{dF}{dy} - (p^2 - \nu)F = 0. \quad (13)$$

where we have denoted

$$p = \frac{1}{2}(k+l+1) \quad (14)$$

This equation has the solution [5]

$$F = F(p - \sqrt{\nu}, p + \sqrt{\nu}, l + \frac{3}{2}, y), \quad (15)$$

which depends on the possible values of the parameter  $p$ . From (12) it follows that

$$k = k_l^\pm = \frac{1}{2} \left[ 1 \pm \sqrt{1 + 4 \left( \frac{M^2}{\lambda^2 \omega^2} - \frac{1}{\lambda} l(l+1) \right)} \right]. \quad (16)$$

which means that we have two possibilities, namely

$$p = p_l^\pm = \frac{1}{2}(k_l^\pm + l + 1) \quad (17)$$

Furthermore, we observe that for

$$\nu = (p + n_r)^2, \quad n_r = 0, 1, 2, \dots, \quad (18)$$

$F$  reduces to a polynomial of degree  $n_r$  in  $y$ . According to these results, we can establish the general form of the solutions of (8), which could be square integrable with respect to (9). This is

$$R_{n_r, l}(r) = N_{n_r, l} (1 + \lambda \omega^2 r^2)^{k/2} r^l F(-n_r, 2p + n_r, l + \frac{3}{2}, -\lambda \omega^2 r^2). \quad (19)$$

The radial quantum number,  $n_r$ , and  $l$  can be embedded into the main quantum number  $n = 2n_r + l$ . It is obvious that  $l$  will take the even values from 0 to  $n$  if  $n$  is even and the odd values from 1 to  $n$  for each odd  $n$ . Now by using (18), (12) and (10) we find the general form of the quantization condition

$$E_{n, l}^2 - M^2 \left(1 + \frac{1}{\lambda}\right) = \lambda \omega^2 [1 - (n + k + 1)^2]. \quad (20)$$

which involves the both quantum numbers,  $n$  and  $l$ , since  $k$  depends on  $l$  as it results from (16). How may be chosen its concrete value we shall see in the following.

Let us first take  $\lambda > 0$ . In this case  $r_e = \infty$ , and the solution (19) will be square integrable only if the condition  $n + k + 1 < 0$  is accomplished. This means that we must take  $k = k_l^- < 0$  (and  $p = p_l^-$ ). Therefore, the discrete energy spectrum will have a finite number of levels with  $n$  and  $l$  selected by the pair of conditions

$$n + k_l^- + 1 < 0, \quad l \leq n. \quad (21)$$

From (20) it results that all these levels satisfy

$$E_{n,l} < E_{lim} = \sqrt{\lambda\omega^2 + M^2 \left(1 + \frac{1}{\lambda}\right)}. \quad (22)$$

On the other hand, from (10) we observe that, for  $E \geq E_{lim}$ ,  $\nu$  is negative or zero and then the hypergeometric functions (15) cannot be reduced to polynomials but remain analytic for negative arguments. Therefore, the functions

$$R_{\nu,l}(r) = N_{\nu,s}(1 + \lambda\omega^2 r^2)^{k_l^-/2} r^l F(p_l^- - \sqrt{\nu}, p_l^- + \sqrt{\nu}, l + \frac{3}{2}, -\lambda\omega^2 r^2) \quad (23)$$

can be interpreted as the non-square integrable solutions corresponding to the continuous energy spectrum,  $[E_{lim}, \infty)$ .

In the case of  $\lambda < 0$  the radial domain is finite the particle being confined to the spherical cavity of the radius  $r_e = 1/\omega\sqrt{-\lambda}$ . The polynomial solutions (19) will be square integrable over  $[0, r_e)$  only if they are regular at  $r_e$ . This require the choice  $k = k_l^+$  (and  $p = p_l^+$ ) for which there are no restrictions on the range of  $n$ . Therefore, the discrete spectrum is countable. Moreover, in this case we have no continuous spectrum since the hypergeometric functions (15) generally diverge for  $y \rightarrow 1$  (when  $r \rightarrow r_e$ ). It is interesting to note that for the particular values of the parameters for which we have

$$M^2 \left(1 + \frac{1}{\lambda}\right) = -\lambda\omega^2 \quad (24)$$

the energy levels become

$$E_n = \sqrt{|\lambda|\omega(k_l^+ + n + 1)}, \quad (25)$$

linearly depending on  $n$  but keeping their fine-structure.

Thus we have found the energy spectra and the energy eigenfunctions up to the normalization constants. These show that the models we have studied can not be considered as rotating oscillators because of the energy term involving simultaneously both the quantum numbers  $n$  and  $l$ . Moreover, the space behavior is also different since here the parameter  $k$  of (19) depend on the quantum number  $l$  while in the case of the rotating oscillators this was a constant independent on  $l$ .

However, despite of these differences, there are some similar features. We refer especially to the continuity with respect to  $\lambda$  in  $\lambda = 0$  and to the nonrelativistic limit. Indeed, we can prove that the solutions we have obtained are continuous with respect to  $\lambda$ , as in the case of the rotating oscillators [3]. Based on this property we can calculate the nonrelativistic limit by taking  $\lambda \rightarrow 0$  and, in addition,  $M \gg \omega$  (i.e.  $Mc^2 \gg \hbar\omega$  in usual units). The conclusion is that all the models we have studied here have the same nonrelativistic limit as that of the rotating oscillators, namely the usual isotropic harmonic oscillator.

## References

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